

LINKAGE OF MODULES AND THE SERRE CONDITIONS

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To the memory of Jan Strooker

ABSTRACT. Let R be semiperfect commutative Noetherian ring and C be a semidualizing R -module. The connection of the Serre condition (S_n) on a horizontally linked R -module of finite G_C -dimension with the vanishing of certain cohomology modules of its linked module is discussed. As a consequence, it is shown that under some conditions Cohen-Macaulayness is preserved under horizontal linkage.

1. INTRODUCTION

The theory of linkage of algebraic varieties was introduced by Peskine and Szpiro [23]. Recall that two ideals \mathfrak{a} and \mathfrak{b} in a Cohen-Macaulay local ring R are said to be linked if there is a regular sequence α in their intersection such that $\mathfrak{a} = (\alpha) : \mathfrak{b}$ and $\mathfrak{b} = (\alpha) : \mathfrak{a}$. The first main theorem in the theory of linkage was due to C. Peskine and L. Szpiro. They proved that over a Gorenstein local ring R with linked ideals \mathfrak{a} and \mathfrak{b} , R/\mathfrak{a} is Cohen-Macaulay if and only if R/\mathfrak{b} is. They also gave a counter-example to show that the above statement is no longer true if the base ring is Cohen-Macaulay but non-Gorenstein. In [24, Theorem 4.1], Schenzel proved that, over a Gorenstein local ring R with maximal ideal \mathfrak{m} , the Serre condition (S_r) for R/\mathfrak{a} is equivalent to the vanishing of the local cohomology groups $H_{\mathfrak{m}}^i(R/\mathfrak{b}) = 0$ for all i , $\dim(R/\mathfrak{b}) - r < i < \dim(R/\mathfrak{b})$, provided \mathfrak{a} and \mathfrak{b} are linked by a Gorenstein ideal \mathfrak{c} (i.e. $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{a} = \mathfrak{c} :_R \mathfrak{b}$ and $\mathfrak{b} = \mathfrak{c} :_R \mathfrak{a}$, equivalently, the ideals $\mathfrak{a}/\mathfrak{c}$ and $\mathfrak{b}/\mathfrak{c}$ of the ring R/\mathfrak{c} are linked by the zero ideal).

In [22], Martsinkovsky and Strooker generalized the notion of linkage for modules over non-commutative semiperfect Noetherian rings over which any finitely generated module admits a projective cover. They introduced the operator $\lambda = \Omega \text{Tr}$ and showed that ideals \mathfrak{a} and \mathfrak{b} are linked by zero ideal if and only if $R/\mathfrak{a} \cong \lambda(R/\mathfrak{b})$ and $R/\mathfrak{b} \cong \lambda(R/\mathfrak{a})$ [22, Proposition 1].

The present authors, in [8, Theorem 4.2], extended Schenzel's result for any horizontally linked module of finite G -dimension over a more general ground ring, i.e. over a Cohen-Macaulay local ring. More precisely, for a horizontally linked module M of finite Gorenstein dimension over a Cohen-Macaulay local ring R , it is shown that M satisfies the Serre condition (S_n) for some $n > 0$ if and only if $H_{\mathfrak{m}}^i(\lambda M) = 0$ for all i , $\dim R - n < i < \dim R$. In this paper, we continue our study about the effect of the Serre condition on a horizontally linked module and extend Schenzel's result in different directions.

Now we describe the organization of the paper. In Section 2, after preliminary notions and definitions, we generalize a result of Auslander and Bridger [4, Theorem 4.25] for modules in the Auslander class \mathcal{A}_C , with respect to a semidualizing module C (see Theorem 2.12). As a consequence, we give a generalization of Schenzel's result for modules in the Auslander class with

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respect to a semidualizing module. For a semidualizing R -module C and a stable R -module M , it is shown that if $M \in \mathcal{A}_C$ and $\text{id}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in \text{Spec } R$ with $\text{depth } R_{\mathfrak{p}} \leq n - 1$, then M satisfies \tilde{S}_n (i.e. $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\{n, \text{depth } R_{\mathfrak{p}}\}$ for all $\mathfrak{p} \in \text{Supp}_R(M)$) if and only if M is horizontally linked and $\text{Ext}_R^i(\lambda M, C) = 0$ for all i , $0 < i < n$ (see Corollary 2.14).

In Section 3, we study the theory of linkage for maximal Cohen-Macaulay modules (mCM). We present a characterization of a mCM-module whose linked module is also mCM (Theorem 3.2). As a consequence, we obtain the following result for the linkage of ideals. Assume that R is a Cohen-Macaulay local ring of dimension d with canonical module ω_R which is generically Gorenstein so that ω_R is identified with an ideal of R . It is shown that if an ideal I is linked to an ideal J by zero ideal such that $I\omega_R = I \cap \omega_R$ and R/I is Cohen-Macaulay, then R/J is Cohen-Macaulay if and only if $R/I + \omega_R$ is Cohen-Macaulay of dimension $d - 1$ (see Corollary 3.3).

In Section 4, for a semidualizing module C , we study the theory of linkage for modules of finite \mathbf{G}_C -dimension which is inspired by authors pervious work [8] on the theory of linkage for modules of finite \mathbf{G} -dimension. Let R be a Cohen-Macaulay local ring, C a semidualizing R -module and c_1, c_2 two \mathbf{G}_C -Gorenstein ideals. Assume that M_1, M , and M_2 are R -modules such that $M_1 \underset{c_1}{\sim} M \underset{c_2}{\sim} M_2$. In Theorem 4.3, it is shown that if $\mathbf{G}_C\text{-dim}_R(M) < \infty$, then M_1 is Cohen-Macaulay if and only if M_2 is. For a horizontally linked module M of finite \mathbf{G}_C -dimension and a positive integer n such that $\lambda M \in \mathcal{A}_C$ (e.g. $\text{pd}_R(\lambda M) < \infty$), it is shown that λM satisfies \tilde{S}_n if and only if $\text{Ext}_R^i(M, C) = 0$ for all i , $0 < i < n$ (see Theorem 4.6), which generalizes [8, Proposition 2.6] and also can be viewed as a generalization of [24, Theorem 4.1]. As a consequence, for a Cohen-Macaulay local ring R and a horizontally linked module M of finite \mathbf{G}_C -dimension such that $\lambda M \in \mathcal{A}_C$, M is mCM if and only if λM is (see Corollary 4.7). In Corollary 4.5, for a horizontally linked R -module M over a Gorenstein local ring R , we determine the attached primes of the local cohomology module $H_{\mathfrak{m}}^{c(M)}(M)$, in terms of the depth of λM , where $c(M)$ denotes the greatest integer $n (< \dim_R(M))$ such that $H_{\mathfrak{m}}^n(M) \neq 0$. As a consequence, it is shown that $H_{\mathfrak{m}}^{c(M)}(M)$ is finitely generated if and only if $\text{depth}_R(\lambda M) + c(M) = \text{depth } R$ and $\text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) + c(M) > \text{depth } R$ for all $\mathfrak{p} \in \text{NCM}(M) \setminus \{\mathfrak{m}\}$, where $\text{NCM}(M)$ denotes the non-Cohen-Macaulay locus of M (see Theorem 4.9).

In Section 5, we study the theory of linkage for reduced \mathbf{G}_C -perfect module (Definition 5.1). These modules can be viewed as a generalization of the Eilenberg-MacLane module [14]. For a reduced \mathbf{G}_C -perfect module M of \mathbf{G}_C -dimension n over a Cohen-Macaulay local ring R , it is shown in Theorem 5.2 that if $\lambda M \in \mathcal{A}_C$, then $\text{depth}_R(M) + \text{depth}_R(\lambda M) = \text{depth } R + \text{depth}_R(\text{Ext}_R^n(M, C))$ which is a generalization of [8, Theorem 3.3]. We end the paper by determining conditions under which an Eilenberg-MacLane horizontally linked R -module is generalized Cohen-Macaulay (see Corollary 5.4).

Throughout the paper, R is a commutative Noetherian ring and all R -modules M, N, \dots are finite (i.e. finitely generated). Whenever, R is assumed local, its unique maximal ideal is denoted by \mathfrak{m} .

For a finite presentation $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$ of an R -module M , its transpose, $\text{Tr}M$, is defined as $\text{Coker } f^*$, where $(-)^* := \text{Hom}_R(-, R)$, which satisfies in the exact sequence

$$(1.1) \quad 0 \rightarrow M^* \rightarrow P_0^* \rightarrow P_1^* \rightarrow \text{Tr}M \rightarrow 0.$$

Moreover, $\text{Tr}M$ is unique up to projective equivalence. Thus all minimal projective presentations of M represent isomorphic transposes of M . For two R -modules M and N , there exists the following exact sequence

$$(1.2) \quad 0 \longrightarrow \text{Ext}_R^1(\text{Tr}M, N) \longrightarrow M \otimes_R N \xrightarrow{e_M^N} \text{Hom}_R(M^*, N) \longrightarrow \text{Ext}_R^2(\text{Tr}M, N) \longrightarrow 0,$$

where $e_M^N : M \otimes_R N \rightarrow \text{Hom}_R(M^*, N)$ is the evaluation map [4, Proposition 2.6].

The syzygy of a module M , denoted by ΩM , is the kernel of an epimorphism $P \xrightarrow{\alpha} M$, where P is a projective R -module, so that it is unique up to projective equivalence. Thus ΩM is uniquely determined, up to isomorphism, by a projective cover of M .

Martsinkovsky and Strooker [22] generalized the notion of linkage for modules over non-commutative semiperfect Noetherian rings (i.e. finitely generated modules over such rings have projective covers).

Definition 1.1. [22, Definition 3] *Let R be a semiperfect ring. Two R -modules M and N are said to be horizontally linked if $M \cong \lambda N$ and $N \cong \lambda M$. Also, M is called horizontally linked (to λM) if $M \cong \lambda^2 M$.*

Note that a commutative ring R is semiperfect if and only if it is a finite direct product of commutative local rings [19, Theorem 23.11]. A *stable* module is a module with no non-zero projective direct summands. Let R be a semiperfect ring, M a stable R -module and $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ a minimal projective presentation of M . Then $P_0^* \rightarrow P_1^* \rightarrow \text{Tr}M \rightarrow 0$ is a minimal projective presentation of $\text{Tr}M$ [1, Theorem 32.13]. The following induced exact sequences

$$(1.1.1) \quad 0 \longrightarrow M^* \longrightarrow P_0^* \longrightarrow \lambda M \longrightarrow 0,$$

$$(1.1.2) \quad 0 \longrightarrow \lambda M \longrightarrow P_1^* \longrightarrow \text{Tr}M \longrightarrow 0,$$

which will be quoted in this paper.

An R -module M is called a *syzygy module* if it is embedded in a projective R -module. Let i be a positive integer, an R -module M is said to be an i th syzygy if there exists an exact sequence

$$0 \rightarrow M \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_0$$

with the P_0, \dots, P_{i-1} are projective. By convention, every module is a 0th syzygy.

A characterization of a module to be horizontally linked, involving a syzygy property, is given below.

Theorem 1.2. [22, Theorem 2 and Proposition 3] *Let R be a semiperfect ring. An R -module M is horizontally linked if and only if it is stable and $\text{Ext}_R^1(\text{Tr}M, R) = 0$, equivalently M is stable and a syzygy module.*

Definition 1.3. *An R -module C is called a semidualizing module, if the homothety morphism $R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism and $\text{Ext}_R^i(C, C) = 0$ for all $i > 0$.*

Semidualizing modules are initially studied in [10] and [13]. It is clear that R itself is a semidualizing R -module. Over a Cohen-Macaulay local ring R , a canonical module ω_R of R is a semidualizing module with finite injective dimension.

Let C be a semidualizing R -module, M an R -module. Let $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$ be a projective presentation of M . The transpose of M with respect to C , $\text{Tr}_C M$, is defined to be $\text{Coker } f^\nabla$, where $(-)^\nabla := \text{Hom}_R(-, C)$, which satisfies the exact sequence

$$(1.3.1) \quad 0 \rightarrow M^\nabla \rightarrow P_0^\nabla \rightarrow P_1^\nabla \rightarrow \text{Tr}_C M \rightarrow 0.$$

By [10, Proposition 3.1], there exists the following exact sequence

$$(1.3.2) \quad 0 \rightarrow \text{Ext}_R^1(\text{Tr}_C M, C) \rightarrow M \rightarrow M^{\nabla\nabla} \rightarrow \text{Ext}_R^2(\text{Tr}_C M, C) \rightarrow 0.$$

Therefore, one has the following exact sequence (see for example [27, Theorem 2.4])

$$(1.3.3) \quad 0 \rightarrow \text{Ext}_R^1(M, C) \rightarrow \text{Tr}_C M \rightarrow (\text{Tr}_C M)^{\nabla\nabla} \rightarrow \text{Ext}_R^2(M, C) \rightarrow 0.$$

The Gorenstein dimension has been extended to \mathbf{G}_C -dimension by Foxby in [10] and by Golod in [13].

Definition 1.4. *An R -module M is said to have \mathbf{G}_C -dimension zero if M is C -reflexive, i.e. the canonical map $M \rightarrow M^{\nabla\nabla}$ is bijective and $\text{Ext}_R^i(M, C) = 0 = \text{Ext}_R^i(M^\nabla, C)$ for all $i > 0$.*

A \mathbf{G}_C -resolution of an R -module M is a right acyclic complex of \mathbf{G}_C -dimension zero modules whose 0th homology is M . The module M is said to have finite \mathbf{G}_C -dimension, denoted by $\mathbf{G}_C\text{-dim}_R(M)$, if it has a \mathbf{G}_C -resolution of finite length. From the exact sequences (1.3.1) and (1.3.2), it is clear that $\mathbf{G}_C\text{-dim}_R(M) = 0$ if and only if $\text{Ext}_R^i(\text{Tr}_C M, C) = 0 = \text{Ext}_R^i(M, C)$ for all $i > 0$. Hence one has the following conclusion.

Remark 1.5. *Let C be a semidualizing R -module. For an R -module M , $\mathbf{G}_C\text{-dim}_R(M) = 0$ if and only if $\mathbf{G}_C\text{-dim}_R(\text{Tr}_C M) = 0$.*

Proof. It is straightforward by using exact sequences (1.3.1), (1.3.2) and (1.3.3). \square

Note that, over a local ring R , a semidualizing R -module C is a canonical module if and only if $\mathbf{G}_C\text{-dim}_R(M) < \infty$ for all finitely generated R -modules M (see [12, Proposition 1.3]).

Recall that the \mathbf{G}_C -dimension of a module M , if finite, can be expressed as follows.

Theorem 1.6. ([13, 4.8]) *For a semidualizing R -module C and an R -module M of finite \mathbf{G}_C -dimension, the following statements hold true.*

- (i) $\mathbf{G}_C\text{-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, C) \neq 0, i \geq 0\}$,
- (ii) *If R is local, then $\mathbf{G}_C\text{-dim}_R(M) = \text{depth } R - \text{depth}_R(M)$.*

2. LINKAGE AND SERRE CONDITIONS

For ideals \mathfrak{a} and \mathfrak{b} in a Gorenstein local ring R which are linked by a Gorenstein ideal \mathfrak{c} , Schenzel in [24, Theorem 4.1] proved that R/\mathfrak{a} satisfies (S_r) if and only if $H_{\mathfrak{m}}^i(R/\mathfrak{b}) = 0$ for all i , $\dim R/\mathfrak{b} - r < i < \dim R/\mathfrak{b}$. In order to develop this theory for modules, we will first generalize the result [4, Theorem 4.25] of Auslander and Bridger for $\text{Tr}_C M$, the transpose of a module M with respect to a semidualizing module C as in Proposition 2.4. We then express Schenzel's result for modules, over Cohen-Macaulay local rings with canonical module, in two forms as in Proposition 2.6 and Corollary 2.8.

In another direction, we extend [17, Theorem 3.3(i)] of Kawasaki and [18, Theorem 1.11], of Khatami and Yassemi for modules in the Auslander class with respect to a semidualizing module.

Throughout, we fix an R -module C and denote $(-)^{\nabla}$ as the dual functor $(-)^{\nabla} := \text{Hom}_R(-, C)$. Two R -modules M and N are said to be stably equivalent with respect C , denoted $M \approx_C N$, if $C^m \oplus M \cong C^n \oplus N$ for some non-negative integers m and n . We write $M \approx N$ when M and N are stably equivalent with respect R . For $k > 0$, the composition $\mathcal{T}_k := \text{Tr} \Omega^{k-1}$ were introduced by Auslander and Bridger in [4].

Remark 2.1. (i) Let R be a semiperfect ring, M an R -module. Assume that $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ is the minimal projective presentation of M . There exists a commutative diagram

$$\begin{array}{ccccccc} P_0^* \otimes_R C & \longrightarrow & P_1^* \otimes_R C & \longrightarrow & \text{Tr} M \otimes_R C & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & & & \\ P_0^{\nabla} & \longrightarrow & P_1^{\nabla} & \longrightarrow & \text{Tr}_C M & \longrightarrow & 0 \end{array}$$

with exact rows. Therefore, $\text{Tr} M \otimes_R C \cong \text{Tr}_C M$.

(ii) For an R -module M and positive integer n , there is an exact sequence

$$0 \rightarrow \text{Ext}_R^n(M, R) \rightarrow \mathcal{T}_n M \rightarrow \lambda^2 \mathcal{T}_n M \rightarrow 0 \text{ (see [8, Remark 1.9])}.$$

Note that $\lambda^2 \mathcal{T}_n M \approx \Omega \mathcal{T}_{n+1} M$. Hence if $\text{Ext}_R^n(M, R) = 0$, then $\mathcal{T}_n M \approx \Omega \mathcal{T}_{n+1} M$.

The proof of the following lemma is based on the proof of [4, Lemma 3.9].

Lemma 2.2. Let C be a semidualizing R -module. If

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

is an exact sequence of R -modules then there exists a natural long exact sequence

$$0 \rightarrow M_3^{\nabla} \rightarrow M_2^{\nabla} \rightarrow M_1^{\nabla} \rightarrow \text{Tr}_C M_3 \rightarrow \text{Tr}_C M_2 \rightarrow \text{Tr}_C M_1 \rightarrow 0.$$

Let C be a semidualizing R -module, M an R -module. For a generating set $\{f_1, f_2, \dots, f_n\}$ of $M^{\nabla} = \text{Hom}_R(M, C)$, denote $f : M \rightarrow C^n$ as the map (f_1, \dots, f_n) . It follows from (1.3.2) that f is injective if and only if $\text{Ext}_R^1(\text{Tr}_C M, C) = 0$. Note that when f is injection, then there is an exact sequence

$$(2.2.1) \quad 0 \rightarrow M \xrightarrow{f} C^n \rightarrow N \rightarrow 0,$$

where $N = \text{Coker}(f)$. It is easy to see that, in this situations, the exact sequence (2.2.1) is dual exact with respect to $(-)^{\nabla}$ and so $\text{Ext}_R^1(N, C) = 0$. Such an exact sequence is called a *universal pushforward of M with respect to C* .

Definition 2.3. [21] An R -module M is said to satisfy the property \tilde{S}_k if $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\{k, \text{depth } R_{\mathfrak{p}}\}$ for all $\mathfrak{p} \in \text{Spec } R$.

Note that, for a horizontally linked module M over a Cohen-Macaulay local ring R , the properties \tilde{S}_k and (S_k) are identical.

For a positive integer n , a module M is called an n th C -syzygy module if there is an exact sequence $0 \rightarrow M \rightarrow C_1 \rightarrow C_2 \rightarrow \dots \rightarrow C_n$, where $C_i \cong \oplus^{m_i} C$ for some m_i .

Let X be a subset of $\text{Spec } R$. An R -module M is said to be of finite G_C -dimension on X , if $G_{C_{\mathfrak{p}}} \text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in X$. We denote $X^n(R) := \{\mathfrak{p} \in \text{Spec}(R) \mid \text{depth } R_{\mathfrak{p}} \leq n\}$.

Recall that an R -module M is n -torsion free if $\text{Ext}_R^i(\text{Tr}M, R) = 0$ for all $1 \leq i \leq n$. In [4, Theorem 4.25], Auslander and Bridger proved that an R -module M of finite Gorenstein dimension is n -torsion free if and only if M satisfies \tilde{S}_n (See also [21, Theorem 42]). In the following we generalize this result in different directions.

Proposition 2.4. *Let C be a semidualizing R -module and M an R -module. For a positive integer n , consider the following statements.*

- (i) $\text{Ext}_R^i(\text{Tr}_C M, C) = 0$ for all i , $1 \leq i \leq n$.
- (ii) M is an n th C -syzygy module.
- (iii) M satisfies \tilde{S}_n .

Then the following implications hold true.

- (a) $(i) \Rightarrow (ii) \Rightarrow (iii)$.
- (b) If M has finite \mathbf{G}_C -dimension on $X^{n-1}(R)$, then $(iii) \Rightarrow (i)$.

Proof. (a). (i) \Rightarrow (ii) Applying $(-)^{\nabla} := \text{Hom}_R(-, C)$ to a projective resolution $\cdots \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M^{\nabla} \rightarrow 0$ of M^{∇} , implies the complex $0 \rightarrow M^{\nabla\nabla} \rightarrow (P_0)^{\nabla} \rightarrow \cdots \rightarrow (P_{n-2})^{\nabla} \rightarrow (P_{n-1})^{\nabla}$. Note that the exact sequence (1.3.2) implies that M is embedded in $M^{\nabla\nabla}$ if $n = 1$ and $M \cong M^{\nabla\nabla}$ if $n > 1$. Therefore M is always 1st C -syzygy and, for $n = 2$, M is 2nd C -syzygy. Assuming $n > 2$ implies $\text{Ext}_R^i(\text{Tr}_C M, C) \cong \text{Ext}_R^{i-2}(M^{\nabla}, C)$ for all i , $2 < i \leq n$, by the exact sequence (1.3.1). Therefore the complex $0 \rightarrow M^{\nabla\nabla} \rightarrow (P_0)^{\nabla} \rightarrow \cdots \rightarrow (P_{n-2})^{\nabla} \rightarrow (P_{n-1})^{\nabla}$ is exact, i.e. M is an n th C -syzygy.

(ii) \Rightarrow (iii). By assumption there is an exact sequence $0 \rightarrow M \rightarrow C_1 \rightarrow \cdots \rightarrow C_n$, where $C_i = C^{l_i}$ for some positive integer l_i , $i = 1, \dots, n$. For any $\mathfrak{p} \in \text{Spec } R$, one has $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\{\text{depth}_{R_{\mathfrak{p}}}(C_n)_{\mathfrak{p}}, n\}$. By [13, page 63(1)], $\text{depth}_{R_{\mathfrak{p}}}(C_n)_{\mathfrak{p}} = \text{depth}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}) = \text{depth } R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec } R$ and so we get the result.

(b). We argue by induction on n . If $n = 1$ then, by Theorem 1.6, M is of \mathbf{G}_C -dimension zero on $X^0(R)$ and so, by Remark 1.5, $\text{Tr}_C M$ is of \mathbf{G}_C -dimension zero on $X^0(R)$. Hence $\text{Ext}_R^1(\text{Tr}_C M, C)_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in X^0(R)$ by Theorem 1.6(i). It is enough to show that $\text{Ass}_R(\text{Ext}_R^1(\text{Tr}_C M, C)) = \emptyset$. Assume contrarily that $\mathfrak{p} \in \text{Ass}_R(\text{Ext}_R^1(\text{Tr}_C M, C))$. By the exact sequence (1.3.2), $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$. As M satisfies \tilde{S}_1 , $\mathfrak{p} \in X^0(R)$, which is a contradiction.

Now, let $n > 1$. By Theorem 1.6, $\mathbf{G}_{C_{\mathfrak{p}}}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for all $\mathfrak{p} \in X^{n-1}(R)$. As M satisfies \tilde{S}_{n-1} , M is of \mathbf{G}_C -dimension 0 on $X^{n-1}(R)$ and so $\text{Ext}_R^1(\text{Tr}_C M, C) = 0$. As noted, after Lemma 2.2, there is a universal pushforward

$$(2.4.1) \quad 0 \rightarrow M \rightarrow C^m \rightarrow N \rightarrow 0.$$

of M with respect to C . As $\text{Ext}_R^1(N, C) = 0$, the exact sequence (2.4.1) implies the exact sequence

$$(2.4.2) \quad 0 \rightarrow \text{Tr}_C N \rightarrow C^m \rightarrow \text{Tr}_C M \rightarrow 0,$$

by Lemma 2.2. Note that N has finite \mathbf{G}_C -dimension on $X^{n-1}(R)$. As M satisfies \tilde{S}_n and $\text{Ext}_R^1(N, C) = 0$, it is easy to see that N satisfies \tilde{S}_{n-1} . Induction hypothesis on N , gives that $\text{Ext}_R^i(\text{Tr}_C N, C) = 0$ for all i , $1 \leq i \leq n-1$. Finally, the exact sequence (2.4.2) implies that $\text{Ext}_R^i(\text{Tr}_C M, C) = 0$ for all i , $1 \leq i \leq n$. \square

As an application of Proposition 2.4, we can find the effect of $M \otimes_R \omega_R$ having the Serre condition (S_n) on the vanishing of the local cohomology groups of λM when M is a horizontally linked module over a Cohen-Macaulay local ring R which admits a canonical module ω_R . This study brings, in particular, some generalizations to the result of Schenzel [24, Theorem 4.1]. First we recall the Local Duality Theorem [7, Corollary 3.5.9].

Theorem 2.5. *Let (R, \mathfrak{m}, k) be a Cohen-Macaulay local ring of dimension d with a canonical module ω_R . Then for all finite R -modules M and all integers i there exist natural isomorphisms*

$$H_{\mathfrak{m}}^i(M) \cong \text{Hom}_R(\text{Ext}_R^{d-i}(M, \omega_R), E_R(k)),$$

where $E_R(k)$ is the injective envelope of k .

Proposition 2.6. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R , M a horizontally linked R -module. Suppose that $M \otimes_R \omega_R$ satisfies (S_1) . Then, for a positive integer n , the following statements are equivalent.*

- (i) λM satisfies (S_n) .
- (ii) $H_{\mathfrak{m}}^i(M \otimes_R \omega_R) = 0$ for all i , $d - n < i < d$.

Proof. By writing (1.2), in terms of ω_R , we may consider $\text{Ext}_R^1(\text{Tr}M, \omega_R)$ as a submodule of $M \otimes_R \omega_R$. From the fact that $M \otimes_R \omega_R$ satisfies (S_1) , it follows that $\text{Ass}_R(\text{Ext}_R^1(\text{Tr}M, \omega_R)) \subseteq \text{Min}R$ which gives $\text{Ext}_R^1(\text{Tr}M, \omega_R) = 0$.

Note that, as $\text{Ass}_R(\lambda M) \subseteq \text{Ass}R$, λM satisfies (S_n) if and only if λM satisfies the condition \tilde{S}_n which in turn is equivalent to say $\text{Tr}M$ satisfies \tilde{S}_{n-1} because λM is the first syzygy of $\text{Tr}M$ and $\text{Ext}_R^1(\text{Tr}M, \omega_R) = 0$. Thus, over the Cohen-Macaulay ring R and by Proposition 2.4, the statement (i) is equivalent to

$$(2.6.1) \quad \text{Ext}_R^i(\text{Tr}_{\omega_R}(\text{Tr}M), \omega_R) = 0 \text{ for all } i, 1 \leq i \leq n-1.$$

On the other hand, Remark 2.1(i) shows that $\text{Tr}_{\omega_R}(\text{Tr}M) \cong \text{Tr} \text{Tr}M \otimes_R \omega_R$. By Theorem 1.2, M is stable, and so $M \cong \text{Tr} \text{Tr}M$. Therefore, $\text{Tr}_{\omega_R}(\text{Tr}M) \cong M \otimes_R \omega_R$. Hence λM satisfies (S_n) if and only if $\text{Ext}_R^i(M \otimes_R \omega_R, \omega_R) = 0$ for all i , $1 \leq i \leq n-1$, which is also equivalent to say that $H_{\mathfrak{m}}^i(M \otimes_R \omega_R) = 0$ for all i , $d - n < i < d$, by the Local Duality Theorem 2.5. \square

To achieve another generalization related to [24], we need the following which is analogous to Proposition 2.4.

Proposition 2.7. *Let C be a semidualizing R -module and M an R -module. For a positive integer n , consider the following statements.*

- (i) $\text{Ext}_R^i(\text{Tr}M, C) = 0$ for all i , $1 \leq i \leq n$.
- (ii) $M \otimes_R C$ is an n th C -syzygy.
- (iii) $M \otimes_R C$ satisfies \tilde{S}_n .

Then the following statements hold true.

- (a) $(i) \Rightarrow (ii) \Rightarrow (iii)$.
- (b) If C has finite injective dimension on $X^{n-1}(R)$, then (iii) implies (i).

Proof. Set $N = M \otimes_R C$. Note that as C is semidualizing, there are natural isomorphisms $N^\nabla \cong M^*$ and $N^{\nabla\nabla} \cong M^{*\nabla}$. Therefore, from the exact sequences (1.2) and (1.3.2), one obtains the commutative diagram

$$\begin{array}{ccccccc}
0 \rightarrow \text{Ext}_R^1(\text{Tr}M, C) & \longrightarrow & N & \longrightarrow & \text{Hom}_R(M^*, C) & \longrightarrow & \text{Ext}_R^2(\text{Tr}M, C) \rightarrow 0 \\
\downarrow & & \downarrow \parallel & & \downarrow \parallel & & \downarrow \\
0 \rightarrow \text{Ext}_R^1(\text{Tr}_C(N), C) & \longrightarrow & N & \longrightarrow & (N)^{\nabla\nabla} & \longrightarrow & \text{Ext}_R^2(\text{Tr}_C(N), C) \rightarrow 0
\end{array}$$

with exact rows, and so $\text{Ext}_R^i(\text{Tr}_C(N), C) \cong \text{Ext}_R^i(\text{Tr}M, C)$ for $i = 1, 2$.

It follows from the exact sequences (1.1.1), (1.1.2), and (1.3.1) that

$$\begin{aligned}
\text{Ext}_R^i(\text{Tr}M, C) &\cong \text{Ext}_R^{i-2}(M^*, C) \\
&\cong \text{Ext}_R^{i-2}(N^\nabla, C) \\
&\cong \text{Ext}_R^i(\text{Tr}_C(N), C)
\end{aligned}$$

for all $i > 2$. Now, by replacing M by N in the Proposition 2.4, the assertion follows. \square

Now we are ready to give another generalization of [24, Theorem 4.1].

Corollary 2.8. *Let R be Cohen-Macaulay local ring of dimension d with canonical module ω_R , M a horizontally linked R -module such that $M \otimes_R \omega_R$ satisfies (S_1) . For a positive integer n , the following statements are equivalent.*

- (i) $M \otimes_R \omega_R$ satisfies (S_n) .
- (ii) $\text{H}_m^i(\lambda M) = 0$ for all i , $d - n < i < d$.

Proof. As in the proof of Proposition 2.6, the fact that $M \otimes_R \omega_R$ satisfies (S_1) implies that $\text{Ext}_R^1(\text{Tr}M, \omega_R) = 0$. Now the assertion is clear by Proposition 2.7 and the Local Duality Theorem 2.5. \square

Let R be a Cohen-Macaulay local ring with canonical module ω_R and let M be an R -module of finite projective dimension. In [17, Theorem 3.3(i)], Kawasaki proved that M is Cohen-Macaulay if and only if $M \otimes_R \omega_R$ is. In [18, Theorem 1.11], Khatami and Yassemi generalized this result for modules of finite Gorenstein dimension. We are going to extend these results for modules in the Auslander class with respect to a semidualizing module. These modules were defined by Avramov and Foxby in [10] and [5].

Definition 2.9. *Let C be a semidualizing R -module. The Auslander class with respect to C , denoted \mathcal{A}_C , consists of all R -modules M satisfying the following conditions.*

- (i) *The natural map $\mu : M \longrightarrow \text{Hom}_R(C, M \otimes_R C)$ is an isomorphism;*
- (ii) *$\text{Tor}_i^R(M, C) = 0 = \text{Ext}_R^i(C, M \otimes_R C)$ for all $i > 0$.*

In the following we collect some properties and examples of modules in the Auslander class with respect to a semidualizing module which will be used in the rest of this paper.

Example 2.10. (i) *If any two R -modules in a short exact sequence are in \mathcal{A}_C , then so is the third one [10, Lemma 1.3]. Hence, every module of finite projective dimension is in the Auslander class \mathcal{A}_C .*

- (ii) Over a Cohen-Macaulay local ring R with canonical module ω_R , $M \in \mathcal{A}_{\omega_R}$ if and only if $\text{G-dim}_R(M) < \infty$ [11, Theorem 1].
- (iii) The \mathcal{I}_C -injective dimension of M , denoted $\mathcal{I}_C\text{-id}_R(M)$, is less than or equal to n if and only if there is an exact sequence

$$0 \rightarrow M \rightarrow \text{Hom}_R(C, I^0) \rightarrow \cdots \rightarrow \text{Hom}_R(C, I^n) \rightarrow 0,$$

such that each I^i is an injective R -module [26, Corollary 2.10]. Note that if M has a finite \mathcal{I}_C -injective dimension, then $M \in \mathcal{A}_C$ [26, Corollary 2.9].

Now, we generalize the results [17, Theorem 3.3(i)] and [18, Theorem 1.11].

Lemma 2.11. *Let R be a local ring, C a semidualizing R -module, $n \geq 0$ an integer, and M an R -module. If $M \in \mathcal{A}_C$, then the following statements hold true.*

- (i) $\text{depth}_R(M) = \text{depth}_R(M \otimes_R C)$ and $\dim_R(M) = \dim_R(M \otimes_R C)$;
- (ii) M satisfies (S_n) if and only if $M \otimes_R C$ does;
- (iii) M is Cohen-Macaulay if and only if $M \otimes_R C$ is Cohen-Macaulay.

Proof. As $M \in \mathcal{A}_C$, $M \cong \text{Hom}_R(C, M \otimes_R C)$ and $\text{Ext}_R^i(C, M \otimes_R C) = 0$ for all $i > 0$. It follows from [3, Lemma 4.1] that

$$\text{depth}_R(M) = \text{depth}_R(\text{Hom}_R(C, M \otimes_R C)) = \text{depth}_R(M \otimes_R C).$$

On the other hand, the fact that $\text{Supp}_R(C) = \text{Spec}(R)$ implies

$$\begin{aligned} \text{Ass}_R(M) &= \text{Ass}_R(\text{Hom}_R(C, M \otimes_R C)) \\ &= \text{Ass}_R(M \otimes_R C). \end{aligned}$$

Therefore, $\dim_R(M) = \dim_R(M \otimes_R C)$. Note that $M_{\mathfrak{p}} \in \mathcal{A}_{C_{\mathfrak{p}}}$ for all $\mathfrak{p} \in \text{Supp}_R(M)$ and so the assertion is clear. \square

In the following, we generalize [4, Theorem 4.25] for modules in the Auslander class with respect to a semidualizing module.

Theorem 2.12. *Let C be a semidualizing R -module and M an R -module. Assume that $M \in \mathcal{A}_C$ and that n is a positive integer. Consider the following statements.*

- (i) $\text{Ext}_R^i(\text{Tr}M, R) = 0$ for $1 \leq i \leq n$;
- (ii) $\text{Ext}_R^i(\text{Tr}M, C) = 0$ for $1 \leq i \leq n$;
- (iii) $M \otimes_R C$ satisfies \tilde{S}_n ;
- (iv) M satisfies \tilde{S}_n .

Then we have the following

- (a) $(i) \Rightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv)$;
- (b) If $\text{G-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in X^{n-1}(R)$ (e.g. $\text{id}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in X^{n-1}(R)$), then all the statements (i)-(iv) are equivalent.

Proof. (a) The equivalence of (iii) and (iv) follows from Lemma 2.11.

(i) \Rightarrow (ii). By [4, Theorem 2.8], there exists the following exact sequence

$$\text{Tor}_2^R(\mathcal{T}_i(\text{Tr}M), C) \rightarrow \text{Ext}_R^i(\text{Tr}M, R) \otimes_R C \rightarrow \text{Ext}_R^i(\text{Tr}M, C) \rightarrow \text{Tor}_1^R(\mathcal{T}_{i+1}(\text{Tr}M), C) \rightarrow 0,$$

for all $i > 0$. As $\text{Ext}_R^i(\text{Tr}M, R) = 0$ for $1 \leq i \leq n$, by Remark 2.1(ii),

$$(2.12.2) \quad \mathcal{T}_i(\text{Tr}M) \approx \Omega \mathcal{T}_{i+1}(\text{Tr}M) \text{ for } 1 \leq i \leq n.$$

Note that $\mathcal{T}_1(\text{Tr}M) \approx M$. As $M \in \mathcal{A}_C$, it follows from (2.12.2) and Example 2.10(i) that $\mathcal{T}_i(\text{Tr}M) \in \mathcal{A}_C$ for all $1 \leq i \leq n+1$ and so $\text{Tor}_j^R(\mathcal{T}_i(\text{Tr}M), C) = 0$ for all $j > 0$ and all i , $1 \leq i \leq n+1$. It follows from the exact sequence (2.12.1) that $\text{Ext}_R^i(\text{Tr}M, C) = 0$ for all $1 \leq i \leq n$.

(ii) \Rightarrow (iii) This follows from Proposition 2.7.

(b)(iv) \Rightarrow (i) Follows from Proposition 2.4, by replacing C by R . \square

Note that every module of finite projective dimension is in the Auslander class with respect to C . The following result is an immediate consequence of Theorem 2.12.

Corollary 2.13. *Let C be a semidualizing R -module and M an R -module of finite projective dimension. For a positive integer n , the following are equivalent.*

- (i) M satisfies \tilde{S}_n ;
- (ii) $M \otimes_R C$ satisfies \tilde{S}_n for every semidualizing module C ;
- (iii) $\text{Ext}_R^i(\text{Tr}M, C) = 0$ for $1 \leq i \leq n$ and for every semidualizing module C ;
- (iv) $M \otimes_R C$ satisfies \tilde{S}_n for some semidualizing module C ;
- (v) $\text{Ext}_R^i(\text{Tr}M, C) = 0$ for $1 \leq i \leq n$ and for some semidualizing module C ;

Corollary 2.14. *Let R be a semiperfect ring, C a semidualizing R -module and M a stable R -module. Assume that $M \in \mathcal{A}_C$ and that n is a positive integer. If $\text{G-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in X^{n-1}(R)$ (e.g. $\text{id}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in X^{n-1}(R)$), then the following are equivalent.*

- (i) M satisfies \tilde{S}_n ;
- (ii) M is horizontally linked and $\text{Ext}_R^i(\lambda M, C) = 0$ for $0 < i < n$.

Proof. This is clear by Theorem 2.12 and Theorem 1.2. \square

3. LINKAGE FOR COHEN-MACAULAY MODULES

The first main theorem in the theory of linkage was due to C. Peskine and L. Szpiro. They proved that over a Gorenstein local ring R with linked ideals \mathfrak{a} and \mathfrak{b} , R/\mathfrak{a} is Cohen-Macaulay if and only if R/\mathfrak{b} is. They also gave a counter-example to show that the above statement is no longer true if the base ring is Cohen-Macaulay but non-Gorenstein. Attempts to generalize this theorem lead to several development in linkage theory, especially by C. Huneke and B. Ulrich ([15] and [16]). In the theory of linkage of modules, Martsinkovsky and Strooker generalize Peskine and Szpiro's result for stable modules over Gorenstein local rings [22, Proposition 8]. In this section, we study the relation between the Cohen-Macaulayness of $M \otimes_R \omega_R$ and λM , for a horizontally linked module M over Cohen-Macaulay local ring R with canonical module ω_R . We also present a characterization of a maximal Cohen-Macaulay module (mCM) whose linked module is also mCM.

Theorem 3.1. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R , M a horizontally linked R -module. If $M \otimes_R \omega_R$ satisfies \tilde{S}_1 , then the following statements are equivalent.*

- (i) $M \otimes_R \omega_R$ is maximal Cohen-Macaulay;

- (ii) λM is maximal Cohen-Macaulay;
- (iii) $M \otimes_R \omega_R$ satisfies (S_n) for some n , $n > d - \text{depth}_R(\lambda M)$;
- (iv) λM satisfies (S_n) for some n , $n > d - \text{depth}_R(M \otimes_R \omega_R)$.

Proof. The equivalence of (i) and (ii) is clear by Corollary 2.8. Trivially (i) implies (iii) and (iv).

(iii) \Rightarrow (ii). It follows from Corollary 2.8 that $H_m^i(\lambda M) = 0$ for all i , $d - n < i < d$. On the other hand, $H_m^i(\lambda M) = 0$ for all $i \leq d - n < \text{depth}_R(\lambda M)$ and so λM is maximal Cohen-Macaulay.

(iv) \Rightarrow (i). It follows from Proposition 2.6 that $H_m^i(M \otimes_R \omega_R) = 0$ for all i , $d - n < i < d$. As $\text{depth}_R(M \otimes_R \omega_R) > d - n$, we conclude that $M \otimes_R \omega_R$ is maximal Cohen-Macaulay. \square

Let R be a Cohen-Macaulay local ring with canonical module ω_R . If R is generically Gorenstein, then ω_R can be identified with an ideal of R . For any such identification ω_R is an ideal of height one or equals R (see [7, Proposition 3.3.18]).

Theorem 3.2. *Let R be a non-Gorenstein Cohen-Macaulay local ring of dimension d which admits a canonical module ω_R . Suppose that R is generically Gorenstein and that M is maximal Cohen-Macaulay horizontally linked R -module such that $M \otimes_R \omega_R$ satisfies (S_1) . Then the following statements are equivalent.*

- (i) λM is maximal Cohen-Macaulay.
- (ii) $M/\omega_R M$ is Cohen-Macaulay of dimension $d - 1$.

Proof. As R is generically Gorenstein and it is not Gorenstein, ω_R can be identified with an ideal of height one. The exact sequence $0 \rightarrow \omega_R \rightarrow R \rightarrow R/\omega_R \rightarrow 0$ implies the exact sequence

$$0 \rightarrow \text{Tor}_1^R(M, R/\omega_R) \rightarrow M \otimes_R \omega_R \rightarrow M \rightarrow M \otimes_R R/\omega_R \rightarrow 0.$$

As $M \otimes_R \omega_R$ satisfies (S_1) and R is generically Gorenstein, it follows that $\text{Tor}_1^R(M, R/\omega_R) = 0$ and one has the following exact sequence

$$(3.2.1) \quad 0 \rightarrow M \otimes_R \omega_R \rightarrow M \rightarrow M \otimes_R R/\omega_R \rightarrow 0.$$

As M is maximal Cohen-Macaulay, from the exact sequence (3.2.1), it is clear that $M \otimes_R \omega_R$ is maximal Cohen-Macaulay if and only if $M \otimes_R R/\omega_R$ is Cohen-Macaulay of dimension $d - 1$. Now the assertion is clear by Theorem 3.1. \square

As an immediate consequence, we have the following result.

Corollary 3.3. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R which is not Gorenstein but it is generically Gorenstein. If the ideals I and J are linked by zero ideal such that $I\omega_R = I \cap \omega_R$ and R/I is Cohen-Macaulay, then R/J is Cohen-Macaulay if and only if $R/I + \omega_R$ is Cohen-Macaulay of dimension $d - 1$.*

Proof. Note that ω_R is identified with an ideal of R [7, Proposition 3.3.18]. As $I\omega_R = I \cap \omega_R$, $\text{Tor}_1^R(R/\omega_R, R/I) \cong \frac{I \cap \omega_R}{I\omega_R} = 0$. By similar argument as in the proof of Theorem 3.2, we obtain the exact sequence $0 \rightarrow (R/I) \otimes_R \omega_R \rightarrow R/I$. As R/I is horizontally linked, it is a first syzygy module. Therefore $(R/I) \otimes_R \omega_R$ is a first syzygy and so satisfies (S_1) . Now the assertion is clear by Theorem 3.2. \square

4. LINKAGE OF MODULES OF FINITE \mathbf{G}_C -DIMENSIONS

Throughout the rest of the paper, R is semiperfect ring and C is a semidualizing R -module. We study the theory of linkage of modules which have finite \mathbf{G}_C -dimensions. The connection of the Serre condition (S_n) on a horizontally linked R -module of finite \mathbf{G}_C -dimension with the vanishing of certain cohomology modules of its linked module is discussed.

An R -module M is said to be *linked* to an R -module N , by an ideal \mathfrak{c} of R , if $\mathfrak{c} \subseteq \text{ann}_R(M) \cap \text{ann}_R(N)$ and M and N are horizontally linked as R/\mathfrak{c} -modules. In this situation we denote $M \underset{\mathfrak{c}}{\sim} N$ [22, Definition 4].

Recall that, for an R -module M we always have $\text{grade}_R(M) \leq \mathbf{G}_C\text{-dim}_R(M)$. The module M is called \mathbf{G}_C -perfect if $\text{grade}_R(M) = \mathbf{G}_C\text{-dim}_R(M)$. An R -module M is called \mathbf{G}_C -Gorenstein if it is \mathbf{G}_C -perfect and $\text{Ext}_R^n(M, C)$ is cyclic, where $n = \mathbf{G}_C\text{-dim}_R(M)$. An ideal I is called \mathbf{G}_C -perfect (resp. \mathbf{G}_C -Gorenstein) if R/I is \mathbf{G}_C -perfect (resp. \mathbf{G}_C -Gorenstein) as R -module. Note that if C is a semidualizing R -module and I is a \mathbf{G}_C -Gorenstein ideal of \mathbf{G}_C -dimension n , then $\text{Ext}_R^n(R/I, C) \cong R/I$ (see [13, 10]).

We recall a result of Golod to be used in the following and more in the sequel [13, Proposition 5].

Theorem 4.1. *Let R be a local ring, I a \mathbf{G}_C -perfect ideal and C a semidualizing R -module. Set $K = \text{Ext}_R^{\text{grade}(I)}(R/I, C)$. Then the following statements hold true.*

- (i) K is a semidualizing R/I -module.
- (ii) If M is a R/I -module, then $\mathbf{G}_C\text{-dim}_R(M) < \infty$ if and only if $\mathbf{G}_K\text{-dim}_{R/I}(M) < \infty$, and if these dimensions are finite then $\mathbf{G}_C\text{-dim}_R(M) = \text{grade}(I) + \mathbf{G}_K\text{-dim}_{R/I}(M)$.

We first present a generalization of [24, Theorem 4.1] for modules of finite \mathbf{G}_C -dimension.

Proposition 4.2. *Let R be a Cohen-Macaulay local ring of dimension d . Suppose that M is an R -module of finite \mathbf{G}_C -dimension and that \mathfrak{c} is a \mathbf{G}_C -Gorenstein ideal of R . Assume that M is linked by \mathfrak{c} and that n is a positive integer n . Then the following statements are equivalent.*

- (i) M satisfies (S_n) .
- (ii) $H_{\mathfrak{m}}^i(\lambda_{R/\mathfrak{c}}M) = 0$ for all i , $\dim R/\mathfrak{c} - n < i < \dim R/\mathfrak{c}$.

Proof. As \mathfrak{c} is \mathbf{G}_C -Gorenstein and $\mathbf{G}_C\text{-dim}_R(M) < \infty$, it follows from Theorem 4.1 that $\mathbf{G}\text{-dim}_{R/\mathfrak{c}}(M) < \infty$. Note that R/\mathfrak{c} is a Cohen-Macaulay ring. By [8, Theorem 4.2], M satisfies (S_n) if and only if $H_{\mathfrak{m}}^i(\lambda_{R/\mathfrak{c}}M) \cong H_{\mathfrak{m}/\mathfrak{c}}^i(\lambda_{R/\mathfrak{c}}M) = 0$ for all i , $\dim R/\mathfrak{c} - n < i < \dim R/\mathfrak{c}$. \square

The reduced grade of a module M with respect to a semidualizing C defined as follows

$$\text{r. grade}(M, C) = \inf\{i > 0 \mid \text{Ext}_R^i(M, C) \neq 0\}.$$

We denote by $\text{r. grade}(M)$, the reduced grade of M with respect to R . If $\text{Ext}_R^i(M, C) = 0$ for all $i > 0$, then $\text{r. grade}(M, C) = +\infty$. Note that if M has a finite and positive \mathbf{G}_C -dimension, then $\text{r. grade}(M, C) \leq \mathbf{G}_C\text{-dim}_R(M)$ by Theorem 1.6.

Recall that two R -modules M and N are said to be in the same even linkage class, or evenly linked, if there is a chain of even length of linked modules that starts with M and ends with N . The following result, shows that the condition \tilde{S}_n is preserved under even linkage.

Theorem 4.3. *Let R be a local ring, C a semidualizing R -module and $\mathfrak{c}_1, \mathfrak{c}_2$ two G_C -Gorenstein ideals. Suppose that M_1, M , and M_2 are R -modules such that $M_1 \sim_{\mathfrak{c}_1} M$ and $M \sim_{\mathfrak{c}_2} M_2$. Assume that $G_C\text{-dim}_R(M) < \infty$ and that $n > 0$ is an integer. Then M_1 satisfies \tilde{S}_n if and only if M_2 satisfies \tilde{S}_n . In particular, if R is Cohen-Macaulay then M_1 is Cohen-Macaulay if and only if M_2 is.*

Proof. As \mathfrak{c}_1 and \mathfrak{c}_2 are G_C -Gorenstein ideals and $G_C\text{-dim}_R(M) < \infty$ we have $G\text{-dim}_{R/\mathfrak{c}_1}(M) < \infty$ and $G\text{-dim}_{R/\mathfrak{c}_2}(M) < \infty$ by Theorem 4.1. Note that by [8, Lemma 5.8], we denote the common value of $\text{grade}_R(\mathfrak{c}_1)$ and $\text{grade}_R(\mathfrak{c}_2)$ by k . By [13, Corollary],

$$(4.3.1) \quad \text{Ext}_{R/\mathfrak{c}_1}^i(M, R/\mathfrak{c}_1) \cong \text{Ext}_R^{i+k}(M, C) \cong \text{Ext}_{R/\mathfrak{c}_2}^i(M, R/\mathfrak{c}_2),$$

for all $i > 0$. By (4.3.1), $r.\text{grade}_{R/\mathfrak{c}_1}(M) = r.\text{grade}_{R/\mathfrak{c}_2}(M)$. By [8, Proposition 2.6], $M_1 = \lambda_{R/\mathfrak{c}_1} M$ satisfies \tilde{S}_n if and only if $r.\text{grade}_{R/\mathfrak{c}_1}(M) = r.\text{grade}_{R/\mathfrak{c}_2}(M) \geq n$, and this is equivalent to saying that $M_2 = \lambda_{R/\mathfrak{c}_2} M$ satisfies \tilde{S}_n by using [8, Proposition 2.6] again. \square

In the following, we express the associated primes of the $\text{Ext}_R^{r.\text{grade}(M,C)}(M, C)$ for a horizontally linked module M of finite and positive G_C -dimension in terms of λM , which is a generalization of [8, Lemma 2.1]. For an integer $n > 0$, we denote the compositions $\mathcal{T}_n^C := \text{Tr}_C \Omega^{n-1}$.

Lemma 4.4. *Let M be a horizontally-linked R -module of finite and positive G_C -dimension. Set $n = r.\text{grade}_R(M, C)$. If $\lambda M \in \mathcal{A}_C$ (e.g. $\text{pd}_R(\lambda M) < \infty$), then*

$$\text{Ass}_R(\text{Ext}_R^n(M, C)) = \{\mathfrak{p} \in \text{Spec } R \mid G_{C_p}\text{-dim}_{R_p}(M_p) \neq 0, \text{depth}_{R_p}((\lambda M)_p) = n = r.\text{grade}_{R_p}(M_p, C_p)\}.$$

Proof. First note that λM is a first syzygy of $\text{Tr } M$ and so $\text{Tr } M \in \mathcal{A}_C$ by Example 2.10(i). Hence by Lemma 2.11 and Remark 2.1(i)

$$(4.4.1) \quad \text{depth}_{R_p}((\text{Tr } M)_p) = \text{depth}_{R_p}((\text{Tr } M)_p \otimes_{R_p} C_p) = \text{depth}_{R_p}((\text{Tr } C M)_p),$$

for all $\mathfrak{p} \in \text{Spec } R$. Let $\cdots \rightarrow P_i \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$ be the minimal projective resolution of M . As $\text{Ext}_R^i(M, C) = 0$ for all $0 < i < n$, applying functor $(-)^{\nabla} = \text{Hom}_R(-, C)$ on the minimal projective resolution of M implies the following exact sequences:

$$(4.4.2) \quad 0 \rightarrow \text{Tr } C M \rightarrow (P_2)^{\nabla} \rightarrow \cdots \rightarrow (P_n)^{\nabla} \rightarrow \mathcal{T}_n^C M \rightarrow 0.$$

$$(4.4.3) \quad 0 \rightarrow \text{Ext}_R^n(M, C) \rightarrow \mathcal{T}_n^C M \rightarrow L \rightarrow 0$$

$$(4.4.4) \quad 0 \rightarrow L \rightarrow \bigoplus_{n+1}^m C \rightarrow \mathcal{T}_{n+1}^C M \rightarrow 0$$

By Theorem 1.2, M is a first syzygy. Therefore, if $G_{C_p}\text{-dim}_{R_p}(M_p) \neq 0$ for some $\mathfrak{p} \in \text{Spec } R$, then it follows from Theorem 1.6 that

$$(4.4.5) \quad \begin{aligned} n &\leq r.\text{grade}(M_p, C_p) \leq G_{C_p}\text{-dim}_{R_p}(M_p) \\ &= \text{depth } R_p - \text{depth}_{R_p}(M_p) < \text{depth } R_p. \end{aligned}$$

Assume that $\mathfrak{p} \in \text{Ass}_R(\text{Ext}_R^n(M, C))$ so that $r.\text{grade}_{R_p}(M_p, C_p) = n$ and $G_{C_p}\text{-dim}_{R_p}(M_p) \neq 0$ by Theorem 1.6. It follows from the exact sequence (4.4.3) that $\text{depth}_{R_p}((\mathcal{T}_n^C M)_p) = 0$. Note that $\text{depth}_{R_p}((P_i)^{\nabla}_p) = \text{depth}_{R_p}(C_p) = \text{depth } R_p$ for all i . By localizing the exact sequence (4.4.2) at \mathfrak{p} , we conclude from (4.4.5) that $\text{depth}_{R_p}((\text{Tr } C M)_p) = n - 1$. Hence, by (4.4.1) and (4.4.5), $\text{depth}_{R_p}((\lambda M)_p) = n$.

Now assume that $\mathfrak{p} \in \text{Spec } R$ such that $G_{C_p}\text{-dim}_{R_p}(M_p) \neq 0$ and $\text{depth}_R((\lambda M)_p) = n = r.\text{grade}_{R_p}(M_p, C_p)$. Hence $\text{depth}_{R_p}((\text{Tr } C M)_p) = n - 1$ by (4.4.1) and (4.4.5). By localizing

the exact sequence (4.4.2) at \mathfrak{p} , we conclude from (4.4.5) that $\text{depth}_{R_{\mathfrak{p}}}((\mathcal{T}_n^C M)_{\mathfrak{p}}) = 0$. As $\text{depth}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}) = \text{depth } R_{\mathfrak{p}} > 0$, we conclude from the exact sequence (4.4.4) that $\text{depth}_{R_{\mathfrak{p}}}(L_{\mathfrak{p}}) > 0$. It follows from the exact sequence (4.4.3) that $\text{depth}_{R_{\mathfrak{p}}}(\text{Ext}_R^n(M, C)_{\mathfrak{p}}) = 0$. In other words, $\mathfrak{p} \in \text{Ass}_R(\text{Ext}_R^n(M, C))$. \square

In order to describe the next result, we recall some notations from the literature. An R -module S is called secondary if, for any $r \in R$, the multiplication map $S \xrightarrow{r} S$ is either surjective or nilpotent; in this case $\mathfrak{p} := \sqrt{0 :_R S}$ is a prime ideal and S is called a \mathfrak{p} -secondary module. It is well-known that any Artinian module T is minimally representable as a sum of its secondary submodules $T = S_1 + \cdots + S_n$ such that S_i is a \mathfrak{p}_i -secondary, $1 \leq i \leq n$, for some distinct prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. These primes are said to be attached and are denoted by $\text{Att}_R(T)$ [20]. We denote the non-Cohen-Macaulay locus of M by

$$\text{NCM}(M) := \{\mathfrak{p} \in \text{Spec } R \mid M_{\mathfrak{p}} \text{ is not a Cohen-Macaulay } R_{\mathfrak{p}}\text{-module}\}.$$

For a non-Cohen-Macaulay module M of dimension d over a local ring (R, \mathfrak{m}) , set

$$c(M) = \sup\{i < d \mid H_{\mathfrak{m}}^i(M) \neq 0\}.$$

As an application, one can describe the attached prime ideals of $H_{\mathfrak{m}}^{c(M)}(M)$ for a non-Cohen-Macaulay horizontally linked R -module M .

Corollary 4.5. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R and let M be an R -module which is a horizontally linked non-Cohen-Macaulay. If $\text{G-dim}_R(\lambda M) < \infty$ (e.g. R is Gorenstein), then*

$$\text{Att}_R(H_{\mathfrak{m}}^{c(M)}(M)) = \{\mathfrak{p} \in \text{Spec } R \mid \mathfrak{p} \in \text{NCM}(M), \text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) = d - c(M) = \dim R_{\mathfrak{p}} - c(M_{\mathfrak{p}})\}.$$

Proof. As M is horizontally linked, it is a first syzygy and so $\dim M_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Supp}(M)$. By [25, 3.4], and the Local Duality Theorem 2.5,

$$\text{Att}_R(H_{\mathfrak{m}}^{c(M)}(M)) = \text{Ass}_R(\text{Ext}_R^{\text{r.grade}(M, \omega_R)}(M, \omega_R)).$$

By Example 2.10(ii), $\lambda M \in \mathcal{A}_{\omega_R}$. Now the assertion is clear by Lemma 4.4 and the Local Duality Theorem. \square

In the following result, we investigate the relation between the Serre condition \tilde{S}_n on a horizontally linked module with the vanishing of certain cohomology modules of its linked module.

Theorem 4.6. *Let M be a stable R -module of finite G_C -dimension and $\lambda M \in \mathcal{A}_C$ (e.g. $\text{pd}_R(\lambda M) < \infty$). For an integer $n > 0$, the following statements hold true.*

- (i) M satisfies \tilde{S}_n if and only if M is horizontally linked and $\text{r.grade}_R(\lambda M) \geq n$;
- (ii) If M is horizontally linked, then $\text{r.grade}(M, C) \geq n$ if and only if λM satisfies \tilde{S}_n .

Proof. (i). By Proposition 2.4, M satisfies \tilde{S}_n if and only if $\text{Ext}_R^i(\text{Tr}_C M, C) = 0$ for all $1 \leq i \leq n$. As mentioned in Remark 2.1(i), $\text{Tr}_C M \cong \text{Tr } M \otimes_R C$. Similar to the proof of [2, Lemma 2.2], it follows from adjointness isomorphism

$$\mathbf{R} \text{Hom}_R(\text{Tr } M \otimes^L C, C) \cong \mathbf{R} \text{Hom}_R(\text{Tr } M, \mathbf{R} \text{Hom}_R(C, C)) \cong \mathbf{R} \text{Hom}_R(\text{Tr } M, R),$$

in the derived category of R , that there is a third quadrant spectral sequence

$$E_2^{p,q} = \text{Ext}_R^p(\text{Tor}_q^R(\text{Tr } M, C), C) \Rightarrow \text{Ext}_R^{p+q}(\text{Tr } M, R).$$

By Example 2.10(i), $\mathrm{Tr}M \in \mathcal{A}_C$ and so $\mathrm{Tor}_i^R(\mathrm{Tr}M, C) = 0$ for all $i > 0$. Hence $E_2^{p,q} = 0$ for all $q > 0$. Therefore, the spectral sequence collapses on p -axis and so

$$\mathrm{Ext}_R^i(\mathrm{Tr}_C M, C) \cong \mathrm{Ext}_R^i(\mathrm{Tr}M \otimes_R C, C) \cong \mathrm{Ext}_R^i(\mathrm{Tr}M, R),$$

for all $i \geq 0$. Therefore, M satisfies \tilde{S}_n if and only if $\mathrm{Ext}_R^i(\mathrm{Tr}M, R) = 0$ for all $1 \leq i \leq n$. Now the assertion is clear by Theorem 1.2.

(ii). Let $\cdots \rightarrow P_i \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$ be the minimal projective resolution of M and let $\mathrm{Ext}_R^i(M, C) = 0$ for all $0 < i < n$. Applying the functor $(-)^{\nabla} = \mathrm{Hom}_R(-, C)$ on the minimal projective resolution of M , gives the exact sequence

$$0 \rightarrow \mathrm{Tr}_C M \rightarrow (P_2)^{\nabla} \rightarrow \cdots \rightarrow (P_n)^{\nabla}.$$

Now it is easy to see that $\mathrm{Tr}_C M (\cong \mathrm{Tr}M \otimes_R C)$ satisfies \tilde{S}_{n-1} . As $\mathrm{Tr}M \in \mathcal{A}_C$, by Lemma 2.11, $\mathrm{Tr}M$ satisfies \tilde{S}_{n-1} . Therefore, λM satisfies \tilde{S}_n .

Conversely, assume that λM satisfies \tilde{S}_n . If $\mathrm{G}_C\text{-dim}_R(M) = 0$ then we have nothing to prove. Assume that $\mathrm{G}_C\text{-dim}_R(M) > 0$. Set $k = \mathrm{r.grade}(M, C)$ and suppose that $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Ext}_R^k(M, C))$. By Lemma 4.4, we have $k = \mathrm{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}})$. Also we have $k = \mathrm{r.grade}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, C_{\mathfrak{p}}) \leq \mathrm{G}_{C_{\mathfrak{p}}}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \mathrm{depth}_{R_{\mathfrak{p}}}$. As λM satisfies \tilde{S}_n , it follows $k = \mathrm{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) \geq \min\{n, \mathrm{depth}_{R_{\mathfrak{p}}}\}$ and so $\mathrm{depth}_{R_{\mathfrak{p}}} > n$. Therefore $\mathrm{r.grade}(M, C) \geq n$. \square

Let R be a Gorenstein local ring and M a horizontally linked R -module. In [22, Proposition 8], it is shown that the maximal Cohen-Macaulayness of M and λM are equivalent. As a consequence of Theorem 4.6, we can generalize [22, Proposition 8] as follows.

Corollary 4.7. *Let R be a Cohen-Macaulay local ring of dimension d and M a stable R -module of finite G_C -dimension. If $\lambda M \in \mathcal{A}_C$ (e.g. $\mathcal{I}_C\text{-id}_R(\lambda M) < \infty$), then the following statements are equivalent.*

- (i) M is maximal Cohen-Macaulay;
- (ii) λM is maximal Cohen-Macaulay and M is horizontally linked;
- (iii) λM satisfies (S_n) for some $n > d - \mathrm{depth}_R(M)$ and M is horizontally linked.

Proof. (i) \Rightarrow (ii). It follows from Theorem 1.6 and Remark 1.5 that $\mathrm{Tr}_C M$ is maximal Cohen-Macaulay. As mentioned in Remark 2.1(i), $\mathrm{Tr}M \otimes_R C \cong \mathrm{Tr}_C M$. By Example 2.10(i), $\mathrm{Tr}M \in \mathcal{A}_C$. Now it follows from Lemma 2.11 that $\mathrm{Tr}M$ is maximal Cohen-Macaulay. Therefore λM is maximal Cohen-Macaulay. As we have seen in the proof of Theorem 4.6, $\mathrm{Ext}_R^i(\mathrm{Tr}_C M, C) \cong \mathrm{Ext}_R^i(\mathrm{Tr}M, R)$ for all $i > 0$. As $\mathrm{G}_C\text{-dim}_R(M) = 0$, $\mathrm{Ext}_R^i(\mathrm{Tr}_C M, C) = 0$ for all $i > 0$. Hence $\mathrm{Ext}_R^1(\mathrm{Tr}M, R) = 0$ and so M is horizontally linked by Theorem 1.2.

(ii) \Rightarrow (iii). Assume that M is horizontally linked and that λM is maximal Cohen-Macaulay so that it satisfies (S_n) for all n and we have nothing to prove.

(iii) \Rightarrow (i). By Theorem 4.6, $\mathrm{r.grade}(M, C) \geq n$. Now assume contrarily that M is not maximal Cohen-Macaulay. Hence $\mathrm{G}_C\text{-dim}_R(M) > 0$ and so we obtain the following inequality from Theorem 1.6:

$$n \leq \mathrm{r.grade}(M, C) \leq \mathrm{G}_C\text{-dim}_R(M) = d - \mathrm{depth}_R(M),$$

which is a contradiction. \square

Let \mathfrak{a} be a \mathbf{G}_C -perfect ideal of R , i.e. R/\mathfrak{a} is \mathbf{G}_C -perfect as R -module. By Theorem 4.1, $\text{Ext}_R^{\text{grade}(\mathfrak{a})}(R/\mathfrak{a}, C)$ is a semidualizing R/\mathfrak{a} -module. As an immediate consequence of Corollary 4.7, we have the following result:

Corollary 4.8. *Let R be a Cohen-Macaulay local ring, M an R -module of finite \mathbf{G}_C -dimension linked by a \mathbf{G}_C -perfect ideal \mathfrak{a} . Assume that $\lambda_{R/\mathfrak{a}}M \in \mathcal{A}_K$, where $K := \text{Ext}_R^{\text{grade}(\mathfrak{a})}(R/\mathfrak{a}, C)$. Then M is Cohen-Macaulay if and only if $\lambda_{R/\mathfrak{a}}M$ is so.*

Proof. First note that R/\mathfrak{a} is a Cohen-Macaulay local ring. As M is linked by the ideal \mathfrak{a} , it follows from Theorem 1.2 that M is a first syzygy as an R/\mathfrak{a} -module and so $\dim_{R/\mathfrak{a}}(M) = \dim R/\mathfrak{a} = \dim_R M$. Therefore M is Cohen-Macaulay if and only if it is maximal Cohen-Macaulay R/\mathfrak{a} -module. By Theorem 4.1, $\mathbf{G}_K\text{-dim}_{R/\mathfrak{a}}(M) < \infty$. Now the assertion is clear by Corollary 4.7. \square

In the following result, we characterize when the local cohomology group $H_{\mathfrak{m}}^{c(M)}(M)$ is of finite length in terms a numerical equality and some inequalities, in certain cases.

Theorem 4.9. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R and let M be a horizontally linked non Cohen-Macaulay R -module. If $\mathbf{G}\text{-dim}_R(\lambda M) < \infty$, then the following are equivalent:*

- (i) $H_{\mathfrak{m}}^{c(M)}(M)$ is finitely generated;
- (ii) $\text{depth}_R(\lambda M) + c(M) = d$ and $\text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) + c(M) > d$ for all $\mathfrak{p} \in \text{NCM}(M) \setminus \{\mathfrak{m}\}$.

Proof. By [6, Corollary 7.2.12], $H_{\mathfrak{m}}^{c(M)}(M)$ is finitely generated if and only if $\text{Att}_R(H_{\mathfrak{m}}^{c(M)}(M)) = \{\mathfrak{m}\}$. Now the implication of (ii) \Rightarrow (i) follows from Corollary 4.5.

(i) \Rightarrow (ii). By Corollary 4.5, $\text{depth}_R(\lambda M) + c(M) = d$. Assume contrarily that

$$(4.9.1) \quad \text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) + c(M) \leq d \text{ for some } \mathfrak{p} \in \text{NCM}(M) \setminus \{\mathfrak{m}\}.$$

Note that, by Local Duality Theorem 2.5, $d - c(M) = \text{r.grade}(M, \omega_R)$. By Theorem 4.6(ii), we obtain

$$(4.9.2) \quad \text{r.grade}(M, \omega_R) \leq \text{r.grade}(M_{\mathfrak{p}}, \omega_{R_{\mathfrak{p}}}) \leq \text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}).$$

It follows from (4.9.1), (4.9.2) that $\text{r.grade}(M, \omega_R) = \text{r.grade}(M_{\mathfrak{p}}, \omega_{R_{\mathfrak{p}}}) = \text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}})$.

Hence by Corollary 4.5, $\mathfrak{p} \in \text{Att}_R(H_{\mathfrak{m}}^{c(M)}(M)) = \{\mathfrak{m}\}$, which is a contradiction. \square

For a ring R , set $X^i(R) = \{\mathfrak{p} \in \text{Spec } R \mid \text{depth } R_{\mathfrak{p}} \leq i\}$. We have the following characterization of horizontally linked module of \mathbf{G}_C -dimension zero.

Theorem 4.10. *Let M be a horizontally linked R -module of finite \mathbf{G}_C -dimension and $\lambda M \in \mathcal{A}_C$. Then $\mathbf{G}_C\text{-dim}_R(M) = 0$ if and only if $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) > \text{depth } R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec } R \setminus X^0(R)$.*

Proof. Set $n = \mathbf{G}_C\text{-dim}_R(M)$. Let $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) > \text{depth } R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec } R \setminus X^0(R)$. Assume contrarily that, $n > 0$. By Lemma 4.4, there exists $\mathfrak{p} \in \text{Spec } R$ such that $\mathbf{G}_{C_{\mathfrak{p}}}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$ and $\text{r.grade}(M, C) = \text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}})$. Therefore by Theorem 1.6, we have

$$\begin{aligned} \text{r.grade}(M, C) &> \text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \\ &= \mathbf{G}_{C_{\mathfrak{p}}}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \text{r.grade}(M_{\mathfrak{p}}, C_{\mathfrak{p}}), \end{aligned}$$

which is a contradiction. Therefore $\mathbf{G}_C\text{-dim}_R(M) = 0$.

Conversely assume that, $\mathbf{G}_C\text{-dim}_R(M) = 0$. Assume contrarily that $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) \leq \text{depth } R_{\mathfrak{p}}$ for some $\mathfrak{p} \in \text{Spec } R \setminus X^0(R)$. It follows that $\mathfrak{p} \in \text{Supp}_R(M)$. Note that $\mathbf{G}_{C_{\mathfrak{p}}}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$. Hence $\mathbf{G}_{C_{\mathfrak{p}}}\text{-dim}_{R_{\mathfrak{p}}}(\text{Tr}_{C_{\mathfrak{p}}} M_{\mathfrak{p}}) = 0$ and so $\text{depth}_{R_{\mathfrak{p}}}(\text{Tr}_{C_{\mathfrak{p}}} M_{\mathfrak{p}}) = \text{depth } R_{\mathfrak{p}}$. By Example 2.10(i), $\text{Tr } M \in \mathcal{A}_C$. It follows from Remark 2.1(i) and Lemma 2.11 that $\text{depth}_{R_{\mathfrak{p}}}(\text{Tr}_{C_{\mathfrak{p}}} M_{\mathfrak{p}}) = \text{depth}_{R_{\mathfrak{p}}}(\text{Tr } M_{\mathfrak{p}})$. Therefore $\text{depth}_{R_{\mathfrak{p}}}(\lambda M_{\mathfrak{p}}) \geq \text{depth } R_{\mathfrak{p}}$. Thus we obtain $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$. By Theorem 1.2, M is a first syzygy and so $\text{depth } R_{\mathfrak{p}} = 0$ which is a contradiction. \square

Recall that a module M is said to be *horizontally self-linked* if $M \cong \lambda M$ (see [22, Definition 7]). It is immediate that the above characterization becomes simpler whenever M is horizontally self-linked.

Corollary 4.11. *Let M be a horizontally self-linked R -module of finite \mathbf{G}_C -dimension and $M \in \mathcal{A}_C$. Then $\mathbf{G}_C\text{-dim}_R(M) = 0$ if and only if $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > \frac{1}{2}(\text{depth } R_{\mathfrak{p}})$ for all $\mathfrak{p} \in \text{Spec } R \setminus X^0(R)$.*

Let R be a local ring. For an R -module M , we recall from [9] that $\text{syz}(M)$ denotes the largest integer n for which M is the n th syzygy in a minimal free resolution of an R -module N . In general, for a horizontally linked R -module M of finite and positive \mathbf{G}_C -dimension, one has

$$(4.11.1) \quad \text{r. grade}(\lambda M) \leq \text{syz}(M) \leq \text{depth}_R(M),$$

by [21, Proposition 11] and Theorem 1.2. In the following, we study under which conditions the equality holds in the above inequality.

For an R -module M , set

$$\text{NG}_C(M) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathbf{G}_{C_{\mathfrak{p}}}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}.$$

The following is a generalization of [8, Theorem 2.7].

Theorem 4.12. *Let R be a local ring, M an R -module with $0 < \mathbf{G}_C\text{-dim}_R(M) < \infty$ and $\lambda M \in \mathcal{A}_C$. If M is horizontally linked then the following conditions are equivalent.*

- (i) $\text{depth}_R(M) = \text{syz}(M) = \text{r. grade}(\lambda M)$;
- (ii) $\mathfrak{m} \in \text{Ass}_R(\text{Ext}_R^{\text{r. grade}(\lambda M)}(\lambda M, R))$;
- (iii) $\text{depth}_R(M) \leq \text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$, for each $\mathfrak{p} \in \text{NG}_C(M)$.

Proof. We first note some general facts. Set $t = \text{r. grade}(\lambda M)$. As $\mathbf{G}_C\text{-dim}_R(M) > 0$, by Theorem 4.6(i), $t < \text{depth } R$. By Remark 2.1(ii), there exists the following exact sequence

$$(4.12.1) \quad 0 \rightarrow \text{Ext}_R^t(\lambda M, R) \rightarrow \mathcal{T}_t(\lambda M) \rightarrow \lambda^2 \mathcal{T}_t(\lambda M) \rightarrow 0,$$

and also $\mathcal{T}_i(\lambda M) \cong \lambda^2 \mathcal{T}_i \lambda M \approx \Omega \mathcal{T}_{i+1}(\lambda M)$ for all i , $0 < i < t$. Therefore,

$$(4.12.2) \quad M \cong \lambda^2 M = \Omega \mathcal{T}_1(\lambda M) \approx \Omega^2 \mathcal{T}_2(\lambda M) \approx \cdots \approx \Omega^t \mathcal{T}_t(\lambda M).$$

(i) \Rightarrow (ii) As $\text{depth}_R(M) = t < \text{depth } R$, it is easy to see that $\text{depth}_R(\mathcal{T}_t(\lambda M)) = 0$ by the (4.12.2). From the exact sequence (4.12.1), as $\text{depth}_R(\lambda^2 \mathcal{T}_t(\lambda M)) > 0$, we find that $\text{depth}_R(\text{Ext}_R^t(\lambda M, R)) = 0$ and so $\mathfrak{m} \in \text{Ass}_R(\text{Ext}_R^t(\lambda M, R))$.

(ii) \Rightarrow (i) By the exact sequence (4.12.1), it is obvious that $\text{depth}_R(\mathcal{T}_t(\lambda M)) = 0$. As $t < \text{depth } R$, $\text{depth}_R(M) = t$ by (4.12.2). Now the assertion is clear by the inequality (4.11.1).

(i) \Rightarrow (iii) By Theorem 4.6(i), M satisfies \tilde{S}_t . Hence if $\mathfrak{p} \in \text{NG}_C(M)$ then $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq t = \text{depth}_R(M)$.

(iii) \Rightarrow (i) Let $\mathfrak{p} \in \text{Ass}_R(\text{Ext}_R^t(\lambda M, R))$. Therefore, $\mathfrak{p}R_{\mathfrak{p}} \in \text{Ass}_{R_{\mathfrak{p}}}(\text{Ext}_{R_{\mathfrak{p}}}^t(\lambda_{R_{\mathfrak{p}}} M_{\mathfrak{p}}, R_{\mathfrak{p}}))$ and $\text{r.grade}_{R_{\mathfrak{p}}}(\lambda_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) = t$. By the exact sequence (4.12.1), we have $\text{depth}_{R_{\mathfrak{p}}}(\mathcal{T}_t(\lambda_{R_{\mathfrak{p}}} M_{\mathfrak{p}})) = 0$. If $\text{G}_{C_p}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$, then $\text{G}_{C_p}\text{-dim}_{R_{\mathfrak{p}}}((\text{Tr}_C M)_{\mathfrak{p}}) = 0$ and so $\text{Ext}_R^i(\text{Tr}_C M, C)_{\mathfrak{p}} = 0$ for all $i > 0$. On the other hand, as seen in the proof of Theorem 4.6, we have

$$\text{Ext}_R^i(\text{Tr} M, R) \cong \text{Ext}_R^i(\text{Tr}_C M, C) \text{ for all } i > 0,$$

which leads to a contradiction. Thus $\text{G}_{C_p}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$. By Theorem 4.6(i), M satisfies \tilde{S}_t which implies $t < \text{depth } R_{\mathfrak{p}}$. Now by localizing (4.12.2) at \mathfrak{p} , we conclude that $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = t$. By our assumption, $\text{depth}_R(M) \leq t$. Now the assertion is clear by the inequality (4.11.1). \square

We can express the reduced grade with respect to a semidualizing module of a horizontally linked module in terms of the depth of linked module as follows, which is a generalization of [8, Proposition 2.2].

Theorem 4.13. *Let M be a horizontally-linked R -module of finite G_C -dimension and $\lambda M \in \mathcal{A}_C$. Then*

$$\text{r.grade}(M, C) = \inf\{\text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Spec } R, \text{G}_{C_p}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}.$$

Moreover, $\text{r.grade}(M) \leq \text{r.grade}(M, C)$ and the equality holds if $\text{pd}_R(\lambda M) < \infty$.

Proof. We may assume that $\text{G}_C\text{-dim}_R(M) > 0$. Set $n = \text{r.grade}(M, C)$. By Theorem 4.6, λM satisfies \tilde{S}_n . Hence,

$$(4.13.1) \quad \text{depth}_{R_{\mathfrak{p}}}(\lambda M)_{\mathfrak{p}} \geq \min\{\text{depth } R_{\mathfrak{p}}, n\} \text{ for all } \mathfrak{p} \in \text{Spec } R$$

Now let $\mathfrak{p} \in \text{Spec } R$ with $\text{G}_{C_p}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$ so that $\text{depth } R_{\mathfrak{p}} > 0$. As M is a syzygy, we get $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > 0$. Therefore, we have $n \leq \text{r.grade}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, C_{\mathfrak{p}}) \leq \text{G}_{C_p}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \text{depth } R_{\mathfrak{p}}$. It follows from (4.13.1) that $\text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) \geq n$, and so $n \leq \inf\{\text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Spec } R, \text{G-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}$.

On the other hand, by the Lemma 4.4, if $\mathfrak{p} \in \text{Ass}_R(\text{Ext}_R^n(M, C))$ then $\text{G}_{C_p}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$ and $\text{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) = \text{r.grade}(M, C)$ and so the assertion holds.

As $\lambda M = \Omega \text{Tr} M$, $\text{Tr} M \in \mathcal{A}_C$ by Example 2.10(i). Note that $M \approx \text{Tr} \text{Tr} M$. Hence it is enough to replace M by $\text{Tr} M$ in the Theorem 2.12 for the last part. \square

For a subset X of $\text{Spec } R$, we say that M is of G_C -dimension zero on X , if $\text{G}_{C_p}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ for all \mathfrak{p} in X .

Proposition 4.14. *Let M be a horizontally linked R -module of finite and positive G_C -dimension and $\lambda M \in \mathcal{A}_C$. Set $t_M = \text{r.grade}(M, C) + \text{r.grade}(\lambda M)$, then M is of G_C -dimension zero on $X^{t_M-1}(R)$.*

Proof. Set $n = \text{r.grade}(\lambda M)$. Assume contrarily that $\text{G}_{C_p}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$ for some $\mathfrak{p} \in X^{t_M-1}(R)$. Note that, by Theorem 4.6(i), we have

$$\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\{\text{depth } R_{\mathfrak{p}}, n\}$$

for all $\mathfrak{p} \in \text{Spec } R$. If $\text{depth } R_{\mathfrak{p}} \leq n$, then $\text{G}_{C_p}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$, which is a contradiction. If $\text{depth } R_{\mathfrak{p}} > n$, then $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq n$. Therefore,

$$t_M - n \leq \text{r.grade}(M_{\mathfrak{p}}, C_{\mathfrak{p}}) \leq \text{G}_{C_p}\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq t_M - n - 1,$$

which is a contradiction. \square

5. REDUCED G_C -PERFECT MODULES

Let M be an R -module of finite positive G_C -dimension. The following inequalities are well-known

$$\text{grade}_R(M) \leq \text{r. grade}(M, C) \leq G_C\text{-dim}_R(M).$$

In the literature, M is called G_C -perfect if $\text{grade}_R(M) = G_C\text{-dim}_R(M)$. In this section we are interested in the case where $\text{r. grade}(M, C) = G_C\text{-dim}_R(M)$ which we call M reduced G_C -perfect. So we bring the following definition.

Definition 5.1. Let M be an R -module of finite G_C -dimension, we say that M is *reduced G_C -perfect* if its G_C -dimension is equal to its reduced grade with respect to C , i.e. $\text{r. grade}(M, C) = G_C\text{-dim}_R(M)$.

Note that every reduced G -perfect module has a finite and positive G_C -dimension. It is obvious that $\text{Ext}_R^{\text{r. grade}_R(M, C)}(M, C)$ is the only non-zero module among all $\text{Ext}_R^i(M, C)$ for $i > 0$.

Let R be a Gorenstein local ring. Following [14], an R -module M is said to be an *Eilenberg-MacLane* module, if $H_m^i(M) = 0$ for all $i \neq \text{depth}_R(M), \dim_R(M)$. Thus reduced G_C -perfect modules can be viewed as a generalization of Eilenberg-MacLane modules. The following is a generalization of [8, Theorem 3.3].

Theorem 5.2. *Let R be a Cohen-Macaulay local ring of dimension d . If M is reduced G_C -perfect of G_C -dimension n and $\lambda M \in \mathcal{A}_C$, then*

$$\text{depth}_R(M) + \text{depth}_R(\lambda M) = \text{depth } R + \text{depth}_R(\text{Ext}_R^n(M, C)).$$

Proof. Let $\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$ be the minimal projective resolution of M . As $\text{Ext}_R^i(M, C) = 0$ for $0 < i < n$, we obtain the following exact sequences (as mentioned in (4.72), (4.7.3) and (4.7.4)):

$$(5.2.1) \quad 0 \rightarrow \text{Tr}_C M \rightarrow (P_2)^\nabla \rightarrow \cdots \rightarrow (P_n)^\nabla \rightarrow \mathcal{T}_n^C M \rightarrow 0,$$

$$(5.2.2) \quad 0 \rightarrow \text{Ext}_R^n(M, C) \rightarrow \mathcal{T}_n^C M \rightarrow L \rightarrow 0$$

$$(5.2.3) \quad 0 \rightarrow L \rightarrow \bigoplus^m C \rightarrow \mathcal{T}_{n+1}^C M \rightarrow 0.$$

As $G_C\text{-dim}_R(\Omega^n M) = 0$, $G_C\text{-dim}_R(\mathcal{T}_{n+1}^C M) = 0$ by Remark 1.5, hence $G_C\text{-dim}_R(L) = 0$. In other words, $\text{depth}_R(L) = d$. It follows from the exact sequence (5.2.2) that

$$(5.2.4) \quad \text{depth}_R(\text{Ext}_R^n(M, C)) = \text{depth}_R(\mathcal{T}_n^C M).$$

By [4, Corollary 4.17], $\text{grade}_R(\text{Ext}_R^n(M, C)) \geq n$. Note that

$$\dim_R(\text{Ext}_R^n(M, C)) = d - \text{grade}_R(\text{Ext}_R^n(M, C))$$

over the Cohen-Macaulay local ring R . Therefore we have

$$\begin{aligned} \text{depth}_R(\mathcal{T}_n^C M) &= \text{depth}_R(\text{Ext}_R^n(M, C)) \\ &\leq \dim_R(\text{Ext}_R^n(M, C)) \\ &\leq d - n. \end{aligned}$$

Now by the exact sequence (5.2.1), it is easy to see that

$$(5.2.5) \quad \text{depth}(\text{Tr}_C M) = \text{depth}_R(\mathcal{T}_n^C M) + n - 1.$$

Note that $\mathrm{Tr}_C M \cong \mathrm{Tr} M \otimes_R C$ by Remark 2.1(i). As λM is the first syzygy of $\mathrm{Tr} M$, it follows from Example 2.10(i) that $\mathrm{Tr} M \in \mathcal{A}_C$. By Lemma 2.11, $\mathrm{depth}_R(\mathrm{Tr} M) = \mathrm{depth}(\mathrm{Tr}_C M)$ and so

$$(5.2.6) \quad \mathrm{depth}_R(\lambda M) = \mathrm{depth}(\mathrm{Tr}_C M) + 1.$$

Now the assertion is clear by (5.2.4), (5.2.5), (5.2.6) and Theorem 1.6. \square

Theorem 5.3. *Assume that M is a horizontally linked R -module with finite and positive $\mathrm{G}_C\text{-dim}_R(M) = n$ and that $\lambda M \in \mathcal{A}_C$. Then M is reduced G_C -perfect if and only if λM satisfies \tilde{S}_n .*

Proof. As M has a positive and finite G_C -dimension n , $\mathrm{r.grade}(M, C) \leq \mathrm{G}_C\text{-dim}_R(M) = n$. On the other hand, by Theorem 4.6, λM satisfies \tilde{S}_n if and only if $\mathrm{r.grade}(M, C) \geq n$. Therefore the assertion is obvious. \square

In the following, we present an equivalent condition for an Eilenberg-MacLane horizontally linked R -module in terms of its linked module.

Corollary 5.4. *Assume that R is a Cohen-Macaulay local ring of dimension d and that M is a horizontally-linked R -module with $\mathrm{depth}_R(M) = n < d$. If $\mathrm{G-dim}_R(\lambda M) < \infty$, then M is Eilenberg-MacLane module if and only if λM satisfies \tilde{S}_{d-n} .*

Proof. We may assume that R is complete with the canonical module ω_R . By Example 2.10(ii), $\lambda M \in \mathcal{A}_{\omega_R}$. Now the assertion is clear by Theorem 5.3 and the Local Duality Theorem 2.5. \square

Recall that an R -module M of dimension $d \geq 1$ is called a *generalized Cohen-Macaulay* module if $\ell(H_{\mathfrak{m}}^i(M)) < \infty$ for all i , $0 \leq i \leq d-1$, where ℓ denotes the length. For an R -module M over a local ring R , it is well-known that

$$\mathrm{depth}_R(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\}.$$

Hence, if M is an Eilenberg-MacLane R -module which is not maximal Cohen-Macaulay, then $\mathrm{c}(M) = \mathrm{depth}_R(M)$. We end the paper by the following result which is an immediate consequence of Theorem 4.9.

Corollary 5.5. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R and let M be a horizontally linked R -module which is not Cohen-Macaulay R -module. Assume that M is an Eilenberg-MacLane module and that $\mathrm{G-dim}_R(\lambda M) < \infty$ (e.g. R is Gorenstein). Then M is generalized Cohen-Macaulay if and only if $\mathrm{depth}_{R_{\mathfrak{p}}}((\lambda M)_{\mathfrak{p}}) + \mathrm{depth}_R(M) > d$ for all $\mathfrak{p} \in \mathrm{NCM}(M) \setminus \{\mathfrak{m}\}$ and $\mathrm{depth}_R(\lambda M) + \mathrm{depth}_R(M) = d$.*

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